REVIEWS

Relaxation Methods in Theoretical Physics, Volume II, by R. V. Southwell. Oxford : Clarendon Press, 1956. 274 pp. 55*s.*

It is not often that a scientific treatise is at once a comprehensive account of the subject and a record of much of the professional career of the author. But Sir Richard Southwell has made the subject of relaxational methods peculiarly his own, and the work reviewed here fits the dual description.

Anticipations or hints by early writers usually make it impossible to set precisely the date of invention of an idea. In 1924 Hardy Cross introduced one of the major concepts of the relaxational approach in conjunction with his ' moment distribution ' method for the calculation of the forces in statically indeterminate structures. Thom's device for solving Laplace's equation (1928) differs from what we now call relaxation only by a small yet important technicality. Be this as it may, the concept and use of the relaxational technique as a systematic device for obtaining numerical solutions to physical problems of all descriptions derives from the investigations of Southwell beginning in 1935. Since that time he has been the high priest of relaxation : a faithful worker, an enthusiastic proponent of its virtues, and, in a small way, the founder of a school. The major part of the literature on the subject is by Southwell and by younger men who entered the field under his tutelage.

This, the second volume of *Relaxation Methods in Theoretical Physics*, appears ten years after publication of the first. The two together make a whole, and they have been so treated for the purpose of this review. In the preface to the first volume, Southwell conceived the work to be " of less concern to engineering than to theoretical physics ", in contradistinction to his 1940 volume on *Relaxation Methods in Engineering Science*. Either Southwell has misjudged the situation, or in the meantime the scope of engineering has grown mightily, for most of the problems he deals with are of first-rate significance in engineering. The stated aims of the two volumes are to set out the underlying theory in a coherent and detailed manner, to describe the various techniques of computation proved by experience, and to convey the rather different atitude conferred by the approach to a physical problem through relaxation. All this Southwell has done in a lucid if rather formal manner.

Both volumes are concerned with physical problems of the classical continuum type. With one minor digression, the range of problems is confined to those having two independent variables : either two space coordinates, or one space and one time coordinate. The first volume begins with a statement of certain aspects of the calculus of finite differences, as a preliminary to the reformulation of partial differential equations as algebraic difference equations. About half of this volume is concerned with the numerical solution of Laplace's equation in the plane, including conformal mapping by numerical methods, and with the quasi-plane-potential problem. The first volume closes with ' free surface ' problems involving boundaries or interfaces not initially known. The second volume continues with more difficult problems, many of which were still classified when the first volume appeared. It deals principally with the biharmonic equation; with eigenvalue problems; with non-linear problems of large elastic distortions, viscous fluid motions, and plastic straining; and it concludes with a brief resumé of recent innovations in relaxation.

When a differential equation purporting to describe a physical phenomenon does not yield to analysis, relaxation, with its entirely different outlook, may be used to advantage. The relaxer does not think in terms of independent and dependent variables as such. He fastens attention on the *numerical* values of the wanted quantity at the nodal points of a finite net in the plane (or the space) of the independent variables. Relaxation comprises a systematic method for determining these numerical values so that they satisfy the finite-difference version of the differential equation. Once the problem has been formulated in terms of relaxation operations indicating the steps required to liquidate the residues (or errors) of the finite-difference equation, its numerical solution requires only simple arithmetical calculations.

Southwell quite rightly brings in much physical argument, for this helps the relaxer to get on with his job. But the reviewer is not the first to be irritated and distracted by Southwell's insistence on thinking of so many problems in terms of the analogy of the force-displacement system in a net made up of strings. This does in fact permit one to revive in a rather strained form the original meaning of the term ' relaxation ', that is, the systematic relaxation of artificial constraints at the joints of a redundant structure, until, all such artificial constraints having been relaxed, the solution sought for is at hand. But while the string-net analogue may somehow illuminate for Southwell problems in, say, potential fluid motion, it interferes with clarity of presentation for others.

Relaxation has a brute force quality which makes it possible, at least in principle, to solve numerically any problem that can be quantitatively formulated in terms of algebraic, differential or integral equations. The wielder of the pencil need know no mathematics beyond arithmetic. Nevertheless mathematical skill and physical insight are assets which usually greatly reduce the time required to effect a solution. Above all, the calculator should have the qualities of an accountant—patience, neatness, and an empathy for numbers. While the arithmetical operations are simple enough, the whole process proceeds rapidly only after a certain amount of experience has been gained. On new and exceptionally difficult problems, the circumstances are often such that various bits of analysis must be invented *ad hoc* as the calculations proceed, and the formulator of the problem may easily find himself in the position of doing the tedious arithmetic. After the few weeks or months required to complete his problem, he has become an expert relaxer, but unless he is one of those for whom relaxation possesss a special fascination, he may thenceforth avoid it like the plague.

The present position of relaxation methods so far as the differential equations of mathematical physics are concerned is that it has indeed demonstrated its ability to solve problems before which more orthodox mathematical methods fail. From a practical point of view, these may be classified in two categories. In one class, e.g., Laplace's equation for plane, irrotational motion of an incompressible fluid, the calculations are so straightforward and progress so rapidly toward a solution that the method is perfectly feasible for the engineering office or for almost any circumstance where, say, a day's labour is not excessive. The other class, e.g., the motion of compressible gases at high speed, or the viscous motion of fluids when inertia cannot be ignored, contains problems requiring the labour of weeks or even months, with the frequent application of analytical thinking. Such problems demand a determined and motivated individual who can spend the necessary time without distraction. One need hardly say that such is not likely to be the case in an engineering office. But the solutions of even a few such problems may settle certain crucial questions arising in a research effort, and each problem solved may exemplify some phenomenon in such manner as to enlarge greatly our understanding if not our capacity to deal quantitatively with it.

An enumeration of titles from the " Index of Problems Solved " delineates the range of problems in fluid mechanics that have yielded to relaxational methods :

Conformal transformation of an area into a circle;
Conformal representation of the region external to an airscrew section into the region external to a circle;
Oil pressure-distribution in a plane slider bearing;
Oil pressure- and temperature-distributions in a partial-journal bearing;
Flow of gas through a convergent-divergent nozzle;
Percolation through an earth wall;
Laminar flow through a two-dimensional Borda mouthpiece;
Orifice plate in a circular tube;
Free jet falling under gravity (' waterfall ');
Slow motion of viscous fluid through a two-dimensional nozzle;

Only a fraction of the book is concerned with problems of fluid mechanics, however. Major sections deal with elasticity, plasticity, and vibrations, and a few problems in heat conduction and electricity are discussed.

The final chapter of Volume II, apparently written after the main body went to press, gives brief accounts of two recent ideas, both due to D. N. de G. Allen. The first employs the device of isometric representation on a single sheet of paper to effect a three-dimensional relaxation solution, a feat which would strain the sanity of any one attempting it on multiple leaves of paper. The second converts a ' marching ' type of problem to one of ' jury ' type, thereby making the problem accessible to relaxation methods. The trick here is to effect a substitution of variables which yields an equation of higher order; since the actual number of physical boundary equations is unaltered, one or more boundary conditions in the substituted variable becomes free both in its value and in its location, and the problem expressed in the substituted variable may thus be framed in the jury form. By this device the parabolic differential equations of unsteady heat conduction, for instance, may be solved by relaxation.

Although Southwell's single-minded enthusiasm might lead one to believe otherwise, there are few circumstances in which a relaxation solution would be preferred to an analytical solution. Questions of computation time aside, the analytical solution possesses two decisive advantages. A single analytical solution in closed form actually comprises many solutions by reason of the parametric constants which appear, And, of great importance to the applied physicist and engineer, it exhibits in condensed form certain patterns of physical behaviour which no mere collection of sheets of relaxation solutions, each with its own array of numbers, can possibly evoke.

Southwell's presentation of the subject incorporates a balance between the physical, the mathematical, and the computational sides. Much of the subject matter is discussed in reference to specific physical problems, and many complete numerical solutions are displayed elegantly and clearly. However, one looks in vain for any critical assessment of the physical models employed, or for meaningful interpretations of the results. Southwell considers the problem ' solved ' when the numerical solution is completed. This should be taken as an observation rather than a criticism, for the book is really concerned with relaxation *methods* and not with physical problems. The mathematical aspects are given their full share of attention, with the thought that this would be one of the more useful items for those who might wish to apply relaxation to an as yet unsolved problem. The various labour-saving tricks of computation and schemes for economically organizing the work are discussed to the point where further elaboration would not be as useful as actual practice on the part of the reader.

It is disappointing that, in a book concerned with numerical methods of solution, there is no discussion of the relationship of relaxation methods to the recent flourishing progress in high-speed machine computation. Relaxation calculations for differential equations with two or three independent variables pose difficulties for a high-speed machine because information must be remembered at a great many net points. Several questions come to mind. Having regard for the special characteristics of digital computers, how can the relaxation procedure be adapted to use best the machine's limited memory but great speed ? Is it feasible or desirable to use many fewer net points in conjunction with higher-order terms in the Taylor-series expansion about a central point ? Is it better for the machine to search for the largest residual or to relax the residuals at the net points in some fixed geometrical sequence? The reviewer would have welcomed reflections along these lines.

Finally, brief mention may be made of two other books on relaxation methods*. These are both much shorter books dealing with the whole of the subject in the sense that algebraic equations and ordinary differential equations, as well as partial differential equations and eigenvalue problems, are treated. Shaw's book omits all discussion of physical problems, and is focussed on the mathematical theory underlying relaxation and on the computational procedures. Allen's book is more akin to Southwell's in its description and illustration of the subject through the medium of physical problems.

For the novice to the subject, Allen's book is, in the opinion of this reviewer, the easiest and best organized introduction to the subject. Southwell's book is the definitive treatise on the subject, giving extensive details of the physical, mathematical, and computational sides, and constituting a compendium of the significant types of problems accessible to relaxation. And Shaw's book seems best suited as an introduction for those interested only in the mathematical and computational aspects of relaxation.

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Proceedings of the 8th International Congress on Theoretical and Applied Mechanics (Istanbul, 1952). Published by the Faculty of Science of the University of Istanbul. Vol. 1, 1953; 529 pp. Vol. 2, 1955; 203 pp. £3 15*s*. 0*d*. for the two volumes.

These two volumes providing a record of the 8th International Congress for Applied Mechanics, held at Istanbul in August 1952, are now available in Britain and U.S.A. The first volume contains the list of members of the congress, the scientific programme, the addresses read at the opening and closing sessions, and summaries (a page or two in length) of the 336 papers read at the congress. The second volume contains extended accounts of most of the seven general lectures and ten sectional addresses.

* *An Introduction to Relaxation Methods*, by F. S. Shaw. New York: Dover, 1953. *Relaxation Methods*, by D. N. de G. Allen. New York: McGraw-Hill, 1954.